Journal of Statistical Physics, Vol. 106, Nos. 5/6, March 2002 (© 2002)

Exact Eternal Solutions of the Boltzmann Equation

A. V. Bobylev[1] **and C. Cercignani**[2]

Received May 9, 2001; accepted October 17, 2001

We construct two families of self-similar solutions of the Boltzmann equation in an explicit form. They turn out to be eternal and positive. They do not possess finite energy. Asymptotic properties of the solutions are also studied.

KEY WORDS: Eternal solutions; Boltzmann equation.

1. INTRODUCTION

We consider a class of solutions of the Boltzmann equation which can be constructed starting from a rather peculiar, self-similar solution with infinite energy. In the particular case of Maxwell molecules, a similar problem was considered 25 years ago when Bobylev[(1)] constructed self-similar solutions with finite energy (obeying the same equation as our solution, but with positive rather than negative values of a certain parameter λ). Several authors (see refs. 2 and 3 for a review) considered these solutions in more detail and finally it was proved by Barnsley and Cornille[(4)] (for the simplest case) that all these solutions, except the so-called "BKW-mode," do not correspond to positive distribution functions. Thus the solutions do not appear to be useful for applications.

We were led to considering these solutions again (but for $\lambda < 0$) by the interesting question of extending the solution for the structure of an infinitely strong shock wave from the case of hard spheres (or cutoff potentials)[(5–7)] to that of molecules interacting at distance. This connection will be discussed in a forthcoming paper,[(8)] together with other aspects of the problem of finding similarity solutions.

[1] Division of Engineering Sciences, Physics and Mathematics, Karlstad University, Karlstad, Sweden.

[2] Dipartimento di Matematica, Politecnico di Milano, Milano, Italy; e-mail: carcer@mate.polimi.it

0022-4715/02/0300-1019/0

The goal of this paper is to present some of the solutions in an explicit form. The paper is organized as follows. In Section 2 we define these solutions and in Section 3 we indicate how to obtain them in the case of pseudo-Maxwell molecules with isotropic scattering. In Section 4 we obtain our first example of exact solution and show its positivity, at variance with the previous case of positive λ. We also show that the solution is eternal, i.e., it exists from $t=-\infty$ to $t=+\infty$. We recall that a recent conjecture states that the only eternal solutions of the Boltzmann equation are Maxwellians.[(9)] In Section 5 we obtain another eternal, positive solution, study its asymptotics and discuss possible generalizations.

Multiplying any solution in the Fourier space by a Maxwellian, we obtain a family of eternal solutions which tend to this Maxwellian at $t=-\infty$. This also makes the distribution function infinitely smooth. Thus our two self-similar solutions originate two one-parameter families of solutions of this kind.

2. THE EQUATION FOR SELF-SIMILAR SOLUTIONS

Let $f(\mathbf{v}, t)$ (where $\mathbf{v} \in \mathfrak{R}^3$ and $t \in \mathfrak{R}_+$ are the velocity and time variables) be a distribution function satisfying the homogeneous Boltzmann equation for Maxwell's molecules:

$$f_t = \int_{\mathfrak{R}^3 \times S^2} d\mathbf{w}\, d\mathbf{n}\, g\left(\frac{\mathbf{V}\cdot\mathbf{n}}{|\mathbf{V}|}\right)[f(\mathbf{v}')\, f(\mathbf{w}') - f(\mathbf{v})\, f(\mathbf{w})] \tag{2.1}$$

where

$$\mathbf{V} = \mathbf{v} - \mathbf{w}, \qquad \mathbf{v}' = \tfrac{1}{2}(\mathbf{v}+\mathbf{w}+|\mathbf{V}|\,\mathbf{n}), \qquad \mathbf{w}' = \tfrac{1}{2}(\mathbf{v}+\mathbf{w}-|\mathbf{V}|\,\mathbf{n}), \qquad \mathbf{n} \in S^2$$

and $g(\cos\theta)$ denotes the scattering cross section multiplied by $|\mathbf{V}|$. For simplicity we do not indicate the time dependence in the collision term.

Performing the Fourier transform

$$\hat{f}(\mathbf{k}) = \int_{\mathfrak{R}^3} d\mathbf{v}\, f(\mathbf{v})\, e^{-i\mathbf{k}\cdot\mathbf{v}}, \qquad \mathbf{k} \in \mathfrak{R}^3 \tag{2.2}$$

we obtain[(3)]

$$\hat{f}_t = \int_{S^2} d\mathbf{n}\, g\left(\frac{\mathbf{k}\cdot\mathbf{n}}{|\mathbf{k}|}\right)[\hat{f}(\mathbf{k}_+)\, \hat{f}(\mathbf{k}_-) - \hat{f}(\mathbf{0})\, \hat{f}(\mathbf{k})] \tag{2.3}$$

where

$$\mathbf{k}_{\pm} = \tfrac{1}{2}(\mathbf{k} \pm |\mathbf{k}|\, \mathbf{n}), \qquad \mathbf{n} \in S^2$$

We consider the simplest class of solutions

$$\hat{f}(\mathbf{v}, t) = \phi(x, t), \qquad x = |\mathbf{k}|^2/2 \tag{2.4}$$

corresponding to isotropic distribution functions $f(t, |\mathbf{v}|)$ in (2.1). Then the equation for $\phi(x, t)$ reads

$$\phi_t = \int_0^1 ds\, G(s)[\phi(sx)\, \phi((1-s)\, x) - \phi(0)\, \phi(x)] \tag{2.5}$$

where

$$G(s) = 4\pi g(1-2s), \qquad 0 \leqslant s \leqslant 1 \tag{2.6}$$

Equation (2.5) has the following relevant properties:

(A) mass and energy conservation laws

$$\frac{d}{dt}\phi(0, t) = \frac{d}{dt}\phi'(0, t) = 0 \tag{2.7}$$

where the prime denotes differentiation with respect to x;

(B) invariance under the transformations

$$\tilde{\phi}(x, t) = e^{-\alpha x}\phi(\beta x, t+\gamma) \tag{2.8}$$

Two essentially different cases may be considered. In the first case (the usual one) the solutions have a finite energy:

$$\phi'(0, t) = \tfrac{1}{3}\int_{\mathfrak{R}^3} d\mathbf{v}\, f(\mathbf{v})\, |\mathbf{v}|^2 < \infty \tag{2.9}$$

In the second case the solutions have infinite energy:

$$\phi'(0, t) = \tfrac{1}{3}\int_{\mathfrak{R}^3} d\mathbf{v}\, f(\mathbf{v})\, |\mathbf{v}|^2 = \infty \tag{2.10}$$

We remark that all the terms in the Boltzmann equation (2.1) are well defined for both cases; energy conservation, however, does not make sense in the second case.

Equation (2.3) obviously admits a class of self-similar solutions

$$\hat{f}(\mathbf{k}, t) = \hat{F}(\mathbf{k}e^{-\lambda t}), \qquad \lambda = \text{const.} \tag{2.11}$$

formally corresponding to a class of functions

$$f(\mathbf{v}, t) = e^{3\lambda t} F(\mathbf{v}e^{\lambda t}) \tag{2.12}$$

satisfying the Boltzmann equation (2.1). These solutions contradict energy conservation since formally

$$\int_{\mathfrak{R}^3} d\mathbf{v}\, f(\mathbf{v}, t)\, |\mathbf{v}|^2 = e^{-2\lambda t} \int_{\mathfrak{R}^3} d\mathbf{v}\, F(\mathbf{v})\, |\mathbf{v}|^2 \neq \text{const.} \tag{2.13}$$

The contradiction, however, disappears in two cases:

$$(1) \quad \int_{\mathfrak{R}^3} d\mathbf{v}\, f(\mathbf{v})\, |\mathbf{v}|^2 = 0, \qquad \text{or} \quad (2) \quad \int_{\mathfrak{R}^3} d\mathbf{v}\, f(\mathbf{v})\, |\mathbf{v}|^2 = \infty \tag{2.14}$$

The first case obviously corresponds to 'non-physical' solutions (except for a trivial solution $F = \delta(\mathbf{v})$) since it violates the positivity condition, $f \geqslant 0$. This does not mean that the self-similar solutions (2.11) are useless. One can easily extend a solution of the class (2.11) to a two-parameter family of solutions (for each admissible value of λ):

$$\hat{f}_{\alpha,\lambda}(\mathbf{k}, t) = e^{-\alpha |\mathbf{k}|^2} \hat{F}(\beta \mathbf{k} e^{-\lambda t}) \tag{2.15}$$

where the parameter β is not indicated as a label because it is usually taken to be unity, without much loss of generality. This function is associated with a positive energy, provided $\alpha > 0$. Moreover it describes the relaxation to a Maxwellian distribution as $t \to \infty$, provided $\lambda > 0$ and $F(0) > 0$. Solutions of this kind were first considered by one of the authors.[1] It is well-known that the simplest solution of the form (2.15) is given by

$$\hat{F}(\mathbf{k}) = (1 + \tfrac{1}{2}\, |\mathbf{k}|^2)\, e^{-\frac{1}{2}|\mathbf{k}|^2} \tag{2.16}$$

with

$$\lambda = \frac{\pi}{4} \int_{-1}^{1} d\mu\, g(\mu)(1 - \mu^2) \tag{2.17}$$

Then the corresponding distribution function (inverse Fourier transform)

$$f_{1/2,\lambda}(\mathbf{v}, t) = (2\pi\Theta)^{-3/2}\, e^{-|\mathbf{v}|^2/(2\Theta)} \left[1 + \frac{1-\Theta}{3\Theta} \left(\frac{|\mathbf{v}|^2}{\Theta} - 3 \right) \right], \qquad \Theta = 1 - e^{-\lambda t} \tag{2.18}$$

is positive for sufficiently large t. This solution was first found in 1968 by Krupp,[10] who unfortunately never published his results. Therefore the solution was rediscovered independently by one of the authors[11] and by Krook and Wu[12] and first published in the years 1975–1976. This solution still remains the only known explicit solution for the Boltzmann equation.

It is also known that there exists a countable set of positive values λ in (2.15) corresponding to entire analytic functions $\hat{F}(|\mathbf{k}|^2)$ such that $|\hat{F}(|\mathbf{k}|^2)| \leqslant A \exp(B\,|\mathbf{k}|^2)$, for some $A, B > 0$.[1,3] Then the equality (2.15) with $\alpha > B$ leads to a true solution of the Boltzmann equation (2.1). It was never proved, however, that the corresponding "distribution functions" $f_{\alpha,\lambda}(\mathbf{v}, t)$ are positive; rather, some examples show that this is not the case (see the review papers of refs. 2–4 for details). However, as we shall see below, these positive solutions do exist for negative values of λ. Moreover, some of the solutions can be constructed in an explicit form.

3. THE SPECIAL CASE $g(\mu) = $ CONST.: USE OF THE LAPLACE TRANSFORM

Our aim in this case is to consider the case $\lambda < 0$ in (2.11), (2.12) and construct some new explicit solutions for the particular case $g = \text{const.}$ in (2.1) (pseudo-Maxwell molecules with isotropic scattering law). Without any loss of generality, we assume $G(s) = 1$, $\phi(0) = 1$ in Eq. (2.5) and look for the self-similar solutions:

$$\phi(x, t) = \psi(xe^{-at}), \qquad \psi(0) = 1, \qquad a = 2\lambda \tag{3.1}$$

Then the equation for $\psi(x)$ reads:

$$-ax\psi_x = \frac{1}{x} \int_0^x dy\, \psi(y)\, \psi(x-y) - \psi(x) \tag{3.2}$$

If we assume that

$$\psi(x) = 1 + O(x^\theta), \qquad \theta > 0, \quad x \to 0 \tag{3.3}$$

then

$$a = a(\theta) = \frac{\theta - 1}{\theta(1+\theta)} \tag{3.4}$$

One can easily verify that Eq. (3.1) has a special solution (see (2.16))

$$\psi(x) = (1 + \gamma x)\, e^{-\gamma x}, \qquad \gamma = \text{const.} \tag{3.5}$$

corresponding to the degenerate "eigenvalue" (3.4)

$$a(2) = a(3) = \tfrac{1}{6} \tag{3.6}$$

Equation (3.2) can be obviously simplified by using the Laplace transform

$$u(z) = \int_0^{\infty} dx\, \psi(x)\, e^{-zx} \tag{3.7}$$

satisfying:

$$-a(\theta)(zu)'' - u' = u^2, \qquad zu(z) \to_{z\to\infty} 1 \tag{3.8}$$

or, equivalently, in terms of $y = zu(z)$:

$$-a(\theta)\, z^2 y'' - zy' + y(1-y) = 0, \qquad y(z) \to_{z\to\infty} 1 \tag{3.9}$$

It is convenient to denote a solution of (3.9) by

$$y = y(z;\theta), \qquad a = a(\theta) = \frac{\theta - 1}{\theta(1+\theta)} \tag{3.10}$$

Eq. (3.9) is obviously invariant under the scaling transformation $z \to \gamma z$, $\gamma = \text{const}$. It admits, however, another class of invariance under a transformation connecting solutions with different value of θ, as shown by the following

Lemma 3.1. If $y(z;\theta)$ is a solution of Eq. (3.9) with $\theta > 1$, then a solution $Y(z;\tilde{\theta})$ of the same equation with $\tilde{\theta} < 1$ is given by

$$Y\left(z; \frac{1}{\theta}\right) = 1 - y(z^{-\theta};\theta) \tag{3.11}$$

Proof. Let us transform both the dependent and independent variables in (3.9)

$$Y = 1 - y, \qquad t = z^{\alpha}$$

A short, elementary calculation leads to

$$-bt^2 Y'' - tY' + Y(1-Y) = 0$$

where

$$a = \frac{1+\alpha}{\alpha(1-\alpha)}, \qquad b = -a\alpha^2 = -\frac{\alpha(1+\alpha)}{1-\alpha}$$

Then if $a(\theta)$ is given by (3.4), we have

$$\alpha = -\theta, \qquad b = -a(\theta)\,\theta^2 = a\left(\frac{1}{\theta}\right)$$

and the lemma is proved, because we have connected a solution corresponding to a certain value of θ to another associated with $1/\theta$. We remark that a changes sign in the transformation.

We now apply the lemma to the Laplace transform of (3.5) with $\gamma = -1$. Then

$$y(z;2) = y(z;3) = \frac{z}{z+1} + \frac{z}{(z+1)^2} = 1 - \frac{1}{(1+z)^2}$$

in agreement with (3.5)–(3.10), and

$$Y\left(z;\frac{1}{2}\right) = \frac{1}{(1+z^{-1/2})^2}, \qquad a = -\frac{2}{3}; \tag{3.12}$$

$$Y\left(z;\frac{1}{3}\right) = \frac{1}{(1+z^{-1/3})^2}, \qquad a = -\frac{3}{2} \tag{3.13}$$

Thus we found two new exact solutions to Eq. (3.8). The next step is to investigate the corresponding solutions of the Boltzmann equation.

4. THE FIRST EXACT SOLUTION

The function in (3.12) yields

$$u(z) = \frac{1}{z(1+z^{-1/2})^2} = \frac{1}{z}\sum_0^\infty (-1)^n \frac{n+1}{z^{n/2}}$$

Hence we obtain the following solution of Eq. (3.2) with $a = -2/3$:

$$\psi(z) = \sum_0^\infty (-1)^n \frac{n+1}{\Gamma(n/2+1)}\, x^{n/2}$$

In fact by Laplace-transforming this series term by term we obtain the previous one for $|z| > 1$, and the result follows for any $z \neq 0$ (with $-\pi < \arg z < \pi$) by analytic continuation.

Since $x = |\mathbf{k}|^2/2$, we get

$$\hat{f}(|\mathbf{k}|) = \sum_0^\infty (-1)^n \frac{(n+1)}{\Gamma(n/2+1)} \left(\frac{|\mathbf{k}|}{\sqrt{2}}\right)^n$$

$$= \left(1 + \mathbf{k} \cdot \frac{\partial}{\partial \mathbf{k}}\right) \left[\exp\left(\frac{|\mathbf{k}|^2}{2}\right) - \sum_0^\infty \frac{|\mathbf{k}|^{2n+1}}{2^{n+1/2}\Gamma(n+3/2)} \right]$$

Two formulas from the handbook by Gradshtein and Ryzhik,(13) pp. 938 and 931, i.e.,

$$\Gamma(n+3/2) = \frac{\sqrt{\pi}}{2^{n+1}} (2n+1)!!$$

$$\Phi(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-x^2}\, dx = \frac{2}{\sqrt{\pi}} e^{-t^2} \sum_0^\infty \frac{2^n |t|^{2n+1}}{(2n+1)!!}$$

lead to

$$\sum_0^\infty \frac{|\mathbf{k}|^{2n+1}}{2^{n+1/2}\Gamma(n+3/2)} = e^{\frac{|\mathbf{k}|^2}{2}} \Phi\left(\frac{|\mathbf{k}|}{\sqrt{2}}\right)$$

Since $\Phi(\infty) = 1$, we obtain

$$\hat{f}(|\mathbf{k}|) = \left(1 + \mathbf{k} \cdot \frac{\partial}{\partial \mathbf{k}}\right) \sqrt{\frac{2}{\pi}} \int_{|\mathbf{k}|}^\infty ds\, e^{-\frac{1}{2}(s^2 - |\mathbf{k}|^2)}$$

$$= \sqrt{\frac{2}{\pi}} \left[-|\mathbf{k}| + (1 + |\mathbf{k}|^2) \int_{|\mathbf{k}|}^\infty ds\, e^{-\frac{1}{2}(s^2 - |\mathbf{k}|^2)} \right]$$

$$= \sqrt{\frac{2}{\pi}} \int_{|\mathbf{k}|}^\infty \frac{ds}{s^2} (s^2 - |\mathbf{k}|^2)\, e^{-\frac{1}{2}(s^2 - |\mathbf{k}|^2)}$$

Finally, the change of variables $s \to (s^2 + |\mathbf{k}|^2)^{1/2}$ gives:

$$\hat{f}(|\mathbf{k}|) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{ds\, s^3 e^{-s^2/2}}{(s^2 + |\mathbf{k}|^2)^{3/2}} \tag{4.1}$$

We can now use a well-known representation of the modified Bessel function (ref. 13, p. 959),

$$K_0(sr) = \int_0^\infty \frac{dt \cos tr}{(t^2+s^2)^{1/2}} = \frac{1}{r}\int_0^\infty \frac{dt\, t \sin tr}{(t^2+s^2)^{3/2}}$$

in order to invert the Fourier transform $\hat{f}(|\mathbf{k}|)$ given by (4.1). The distribution function $f = f(|\mathbf{v}|)$ is given by

$$\begin{aligned} f(r) &= \frac{1}{(2\pi)^3}\int_{\mathfrak{R}^3} d\mathbf{k}\, \hat{f}(|\mathbf{k}|)\, e^{i\mathbf{k}\cdot\mathbf{v}} = \frac{1}{2\pi^2 r}\int_0^\infty dt\, \hat{f}(t)\, t \sin(tr) \\ &= \frac{1}{(2\pi^5)^{1/2}}\int_0^\infty ds\, s^3 e^{-s^2/2} K_0(sr), \qquad r = |\mathbf{v}| \end{aligned}$$

Another representation of $K_0(z)$ (ref. 13, p. 959),

$$K_0(z) = \frac{1}{2}\int_0^\infty \frac{dt}{t}\exp\left(-t - \frac{z^2}{4t}\right)$$

leads to

$$f(r) = 2^{-3/2}\pi^{-5/2}\int_0^\infty \frac{dt}{t}\, e^{-t}\int_0^\infty ds\, s^3 e^{-\frac{s^2}{2}(1+\frac{r^2}{2t})} = (2\pi^5)^{-1/2}\int_0^\infty \frac{dt\, te^{-t}}{(t+\frac{r^2}{2})^2}$$

Thus we constructed an exact positive solution of the Boltzmann equation (2.1) with $g(\mu) = (4\pi)^{-1}$. The solution is given by

$$f(\mathbf{v}, t) = e^{-t} F(\mathbf{v}e^{-t/3}) \tag{4.2}$$

where F is given by

$$\begin{aligned} F(\mathbf{v}) &= (2\pi^5)^{-1/2}\int_0^\infty \frac{dt\, te^{-t}}{(t+\frac{r^2}{2})^2} \\ &= (2\pi^5)^{-1/2}\left[-1 + \left(1 + \frac{|\mathbf{v}|^2}{2}\right) e^{|\mathbf{v}|^2/2}\int_{|\mathbf{v}|^2/2}^\infty \frac{ds\, e^{-s}}{s}\right] \end{aligned}$$

The solution is expressed explicitly through elementary functions and the integral exponential function

$$\mathrm{Ei}(-x) = -\int_x^\infty \frac{ds\, e^{-s}}{s}, \qquad x > 0$$

The asymptotic formulas (ref. 13, p. 927)

$$\mathrm{Ei}(-x) = \gamma + \log x + \sum_{0}^{\infty} \frac{(-x)^n}{k \cdot k!} \qquad (\gamma = \text{Euler's constant})$$

$$\mathrm{Ei}(-x) = e^{-x} \left\{ \sum_{0}^{n} \frac{(k-1)!}{(-x)^k} + R_n \right\}, \qquad |R_n| < \frac{n!}{x^n}, \quad x > 0$$

lead to similar series for $F(\mathbf{v})$. In particular:

$$F(\mathbf{v}) \cong \left(\frac{\pi^5}{2}\right) 2^{-1/2} \log \frac{1}{|\mathbf{v}|}, \qquad |\mathbf{v}| \to 0$$

$$F(\mathbf{v}) \cong \frac{1}{2} \left(\frac{2}{\pi}\right)^{5/2} |\mathbf{v}|^{-4}, \qquad |\mathbf{v}| \to \infty$$

The corresponding self-similar solution (4.2) exists for all $t \in \mathfrak{R}$, i.e., it is an example of eternal solution,[9, 14, 15] having the following asymptotic behavior

$$f(\mathbf{v}, t) \cong \left(\frac{\pi^5}{2}\right)^{-1/2} e^{-t} \left[\frac{t}{3} + \log \frac{1}{|\mathbf{v}|} - (\gamma + 1) + O(e^{-\frac{2}{3}t})\right], \qquad t \to +\infty, \quad |\mathbf{v}| > 0$$

$$f(\mathbf{v}, t) \cong \frac{1}{2} \left(\frac{2}{\pi}\right)^{5/2} \frac{e^{-|t|/3}}{|\mathbf{v}|^4} [1 + O(e^{-\frac{2}{3}|t|})], \qquad t \to -\infty, \quad |\mathbf{v}| > 0$$

In the weak sense (on test functions of finite support)

$$f(\mathbf{v}, t) \to_{t \to +\infty} 0 \qquad \text{and} \qquad f(\mathbf{v}, t) \to_{t \to -\infty} \delta(\mathbf{v})$$

It is clear that this asymptotics for $t \to +\infty$ is possible only because the corresponding energy (second moment of $f(\mathbf{v}, t)$) is infinitely large.

Finally we remark that the functions

$$f_\alpha(\mathbf{v}, t) = f * (2\pi\alpha)^{-3/2} e^{-|\mathbf{v}|^2/(2\alpha)}, \qquad \alpha > 0$$

where $*$ denotes the convolution in $\mathfrak{R}^3$, constitute a one-parameter family of smooth eternal solutions of the Boltzmann equation (2.1), so that the logarithmic singularity of $F(\mathbf{v})$ at $\mathbf{v} = 0$ is not so important.

5. THE SECOND EXACT SOLUTION AND GENERALIZATIONS

Let us consider the second function (3.13). Then

$$u(z) = \int_0^\infty dx\, \psi(x)\, e^{-zx} = \frac{1}{z(1+z^{-1/3})^2} = \frac{1}{z} \sum_0^\infty (-1)^n \frac{n+1}{z^{n/3}}, \qquad a = 2\lambda = -\frac{3}{2}$$

where $\psi(x)$ is the solution of Eq. (3.2). By inverting the Laplace transform, we obtain:

$$\psi(x) = \sum_0^\infty (-1)^n \frac{n+1}{\Gamma(1+n/3)}\, x^{n/3}$$

The corresponding solution of the Boltzmann equation (2.1) (provided it exists) reads

$$f(\mathbf{v}, t) = e^{-\frac{9}{4}t} F(\mathbf{v} e^{-\frac{3}{4}t}) \tag{5.1}$$

where

$$\int_{\mathfrak{R}^3} d\mathbf{v}\, F(\mathbf{v})\, e^{-i\mathbf{k}\cdot\mathbf{v}} = \psi\left(\frac{|\mathbf{k}|^2}{2}\right) \tag{5.2}$$

Our main goal in this section is to invert the Fourier transform and to prove that $F(\mathbf{v}) \geqslant 0$. To this end we consider a slightly more general problem.

It is obvious that by applying the original method[3] to the d-dimensional Boltzmann equation ($d \geqslant 2$) we obtain the same Fourier-transformed Boltzmann equation (2.3) with trivial changes ($\mathfrak{R}^3 \to \mathfrak{R}^d$ and $S^2 \to S^{d-1}$). Moreover the function $g(\cos\theta)$ can be always chosen in such a way that leads to the same isotropic equation (2.5) with $\rho(s) = 1$.[2] Hence it is worthwhile to consider the inverse Fourier transform in the general d-dimensional case ($d = 2, 3,...$) in order to describe a class of exact solution of the generalized Boltzmann equation.

On the other hand, the characteristic functions $\psi(|\mathbf{k}|^2/2)$ for both the solution investigated here and the one studied in the previous section can be written in a similar way:

$$\psi\left(\frac{|\mathbf{k}|^2}{2}\right) = \left(\frac{1}{2\alpha}\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{k}} + 1\right) Q_\alpha\left(\frac{|\mathbf{k}|^2}{2}\right), \qquad \alpha = \frac{1}{2}, \frac{1}{3}$$

where

$$Q_\alpha(x) = \sum_0^\infty \frac{(-1)^n}{\Gamma(1+n\alpha)}\, x^{n\alpha} = \mathscr{L}^{-1} \frac{z^{\alpha-1}}{1+z^\alpha} \tag{5.3}$$

is the so-called Mittag–Leffler function.[16] Hence the unknown distribution function $F(\mathbf{v})$ (5.2) reads

$$F_\alpha = \mathscr{F}^{-1}[\psi] = \left(1 - \frac{1}{2\alpha}\frac{\partial}{\partial \mathbf{v}}\cdot \mathbf{v}\right)\mathscr{F}^{-1}\left[Q_\alpha\left(\frac{|\mathbf{k}|^2}{2}\right)\right]$$

where $\mathscr{F}^{-1}$ denotes the inverse Fourier transform. If $\mathbf{v} \in \mathfrak{R}^d$ then

$$\frac{\partial}{\partial \mathbf{v}}\cdot \mathbf{v} = d + \mathbf{v}\cdot\frac{\partial}{\partial \mathbf{v}}$$

therefore

$$F_\alpha = \left[\left(1 - \frac{d}{2\alpha}\right) - \frac{1}{2\alpha}\mathbf{v}\cdot\frac{\partial}{\partial \mathbf{v}}\right]\mathscr{F}^{-1}[Q_\alpha], \qquad d = 2, 3,... \tag{5.4}$$

in the general case. Thus we need to evaluate the integral

$$\mathscr{F}^{-1}[Q_\alpha] = \Phi_\alpha = \frac{1}{(2\pi)^d}\int_{\mathfrak{R}^d} d\mathbf{k}\, Q_\alpha\left(\frac{|\mathbf{k}|^2}{2}\right) e^{i\mathbf{k}\cdot\mathbf{v}} \tag{5.5}$$

A key idea is the following. It is well known in probability theory[16] that for any $0 < \alpha < 1$ there exists a non-negative function $g_\alpha(x)$, $x \geqslant 0$, such that

$$\mathscr{L}[g_\alpha] = \int_0^\infty dx\, g_\alpha(x)\, e^{-zx} = e^{-z^\alpha} \tag{5.6}$$

This function can be used for the following integral representation of Q_α, defined in (5.3):

$$Q_\alpha(x) = \int_0^\infty dy\, g_\alpha(y)\, e^{-(x/y)^\alpha} \tag{5.7}$$

This formula is a consequence of comparing (5.3) (which provides the Laplace transform of Q_α) with the following formula:

$$\int_0^\infty dx\, e^{-zx}\int_0^\infty dy\, g_\alpha(y)\, e^{-(x/y)^\alpha} = \int_0^\infty dt\, e^{-t^\alpha}\int_0^\infty dy\, y g_\alpha(y)\, e^{-zty}$$

$$= \alpha\int_0^\infty dt\, e^{-t^\alpha}(zt)^{\alpha-1}\, e^{-(zt)^\alpha} = \frac{z^{\alpha-1}}{1+z^\alpha}$$

which indicates that the last expression is the Laplace transform of the right hand side of (5.7). Here (5.6) (differentiated with respect to z) has been used.

Equalities (5.6) and (5.7) lead to another integral representation

$$Q_\alpha(x) = \int_0^\infty dy\, g_\alpha(y) \int_0^\infty dz\, g_\alpha(z)\, e^{-x(z/y)} \tag{5.8}$$

which is very convenient to evaluate the inverse Fourier transform (5.5) since

$$(2\pi\Theta)^{-d/2}\, e^{-|\mathbf{v}|^2/(2\Theta)} = \frac{1}{(2\pi)^d} \int_{\Re^d} d\mathbf{k}\, e^{-\Theta\, |\mathbf{k}|^2/2 + i\mathbf{k}\cdot\mathbf{v}}, \qquad \Theta > 0, \quad \mathbf{v} \in \Re^d$$

Hence, taking the Fourier transform of (5.8), with $x = |\mathbf{k}|^2/2$ and using the last relation with $\Theta = z/y$ gives

$$\begin{aligned} \Phi_\alpha(\mathbf{v}) &= \mathscr{F}^{-1}[Q_\alpha] \\ &= \frac{1}{(2\pi)^{d/2}} \int_0^\infty \int_0^\infty dy\, dz \left(\frac{y}{z}\right)^{d/2} g_\alpha(y)\, g_\alpha(z)\, e^{-u(y/z)}, \quad u = |\mathbf{v}|^2/2 \end{aligned} \tag{5.9}$$

The double integral can be easily simplified in the case of an even number of dimensions, $d = 2n$, $n = 1, 2, \ldots$. In fact, differentiating (5.8) n times, we obtain:

$$\Phi_\alpha(\mathbf{v}) = \tilde{\Phi}_\alpha(u) = \frac{1}{(2\pi)^n} \left(\frac{d}{du}\right)^n Q_\alpha(u), \qquad u = |\mathbf{v}|^2/2$$

Then Eq. (5.4) leads to

$$\begin{aligned} F_\alpha(\mathbf{v}) = \tilde{F}_\alpha(u) &= \frac{(-1)^{n+1}}{(2\pi)^n \alpha} \left[(n - \alpha) + u \frac{d}{du} \right] Q_\alpha^{(n)}(u) \\ &= \frac{(-1)^{n+1}}{(2\pi)^n \alpha}\, u^{1+\alpha-n} \frac{d}{du} \left[u^{n-\alpha} Q_\alpha^{(n)}(u) \right] \end{aligned}$$

where $Q_\alpha^{(n)}(u)$ denotes the nth derivative ($n = 1, 2, \ldots$). In the plane case ($n = 1$, $d = 2$), which may be of practical interest, we obtain from (5.7)

$$Q_\alpha^{(1)}(u) = -\alpha u^{\alpha-1} \int_0^\infty dy\, g_\alpha(y)\, y^{-\alpha} e^{-(u/y)^\alpha}$$

Therefore the distribution function is given by

$$\tilde{F}_\alpha(u) = -\frac{u^\alpha}{2\pi}\frac{d}{du}\int_0^\infty dy\, g_\alpha(y)\, y^{-\alpha} e^{-(u/y)^\alpha}$$

$$= \frac{\alpha}{2\pi} u^{2\alpha-1}\int_0^\infty dy\, g_\alpha(y)\, y^{-2\alpha} e^{-(u/y)^\alpha}, \qquad u = |\mathbf{v}|^2/2 \tag{5.10}$$

where $\alpha = 1/2$ or $1/3$ yields the positive solution

$$f_\alpha(\mathbf{v}, t) = e^{-2\lambda_\alpha t}\tilde{F}_a\left(\frac{|\mathbf{v}|^2}{2}e^{-2\lambda_\alpha t}\right) \tag{5.11}$$

with $\lambda_{1/2} = 1/3$ or $\lambda_{1/3} = 3/4$ respectively.

Remark. The results for the plane case could be obtained even more easily. Our approach, however, allows a proof of positivity and works for an arbitrary dimension. It is easy to guess that the explicit solution obtained in Section 4 is connected with the fact that the function

$$g_{1/2} = \frac{1}{2\sqrt{\pi}} x^{-3/2} e^{-1/(4x)} \tag{5.12}$$

is known in an explicit form.[17] The function $g_{1/3}$ is much more complicated.

We shall show that the corresponding solution for the 3d Boltzmann equation (2.1) is also positive and give formulas for its asymptotic behavior.

Let us consider Eq. (5.9) with $d = 3$. It is convenient to work with an arbitrary α ($0 < \alpha < 1$) in order to see the difference between the two cases $\alpha = 1/2$ (first solution) and $\alpha = 1/3$ (second solution). We denote again

$$F_\alpha(\mathbf{v}) = \tilde{F}_\alpha\left(\frac{|\mathbf{v}|^2}{2}\right) \tag{5.13}$$

and omit tildes below. Eqs. (5.4) and (5.9) with $d = 3$ yield

$$F_\alpha(u) = = \frac{u^{\alpha-1/2}}{(2\pi)^{3/2}\,\alpha}\frac{d}{du}\left[u^{3/2-\alpha}\frac{d\mathscr{I}_\alpha(u)}{du}\right] \tag{5.14}$$

where

$$\mathscr{I}_\alpha = \int_0^\infty dy\, g_\alpha(y)\int_0^\infty dz\, g_\alpha(z)\left(\frac{y}{z}\right)^{1/2} e^{-u(y/z)}$$

Noting that

$$\sqrt{r} = \frac{1}{2\Gamma(1/2)} \int_0^\infty \frac{ds}{s^{3/2}} [1 - e^{-rs}], \qquad r = \frac{y}{z}, \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

and using (5.8) we obtain

$$\begin{aligned}\mathscr{I}_\alpha &= \frac{1}{2\Gamma(1/2)} \int_0^\infty \frac{ds}{s^{3/2}} [Q_\alpha(u) - Q_\alpha(s+u)] \\ &= -\frac{1}{\Gamma(1/2)} \int_0^\infty \frac{ds}{\sqrt{s}} Q'_\alpha(s+u)\end{aligned}$$

or, equivalently,

$$\begin{aligned}\mathscr{I}_\alpha &= \frac{\alpha u^{\alpha-1/2}}{\Gamma(1/2)} \int_0^\infty \frac{ds(1+s)^{\alpha-1}}{\sqrt{s}} \int_0^\infty dy\, g_\alpha(y)\, y^{-\alpha} e^{-[u(1+s)/y]^\alpha} \\ &= \frac{u^{\alpha-1/2}}{\Gamma(1/2)} \int_1^\infty \frac{dt}{\sqrt{t^{1/\alpha}-1}} \int_0^\infty dy\, g_\alpha(y)\, y^{-\alpha} e^{-t[u/y]^\alpha}\end{aligned}$$

where we let $t = (1+s)^\alpha$. Then we transform Eq. (5.14) to

$$F_\alpha(u) = \frac{u^{\alpha-1/2}}{(2\pi)^{3/2}\alpha\Gamma(1/2)} \frac{d}{du}\left[u^{3/2-\alpha} \frac{d}{du} (u^{\alpha-1/2} G_\alpha) \right]$$

with

$$G_\alpha(u) = \int_0^\infty dy\, g_\alpha(y)\, y^{-\alpha} \int_1^\infty \frac{dt}{\sqrt{t^{1/\alpha}-1}} e^{-t[u/y]^\alpha} \tag{5.15}$$

The function $F_\alpha(u)$ depends on G'_α only, since

$$\begin{aligned}F_\alpha(u) &= \frac{u^{\alpha-1/2}}{(2\pi)^{3/2}\, \alpha\Gamma(1/2)} \left[\left(\alpha + \frac{1}{2}\right) + u \frac{d}{du} \right] \frac{dG_\alpha}{du} \\ &= \frac{1}{(2\pi)^{3/2}\, \alpha\Gamma(1/2)} \frac{d}{du} \left(u^{\alpha+1/2} \frac{dG_\alpha}{du} \right)\end{aligned}$$

Differentiation of Eq. (5.15) yields

$$G'_\alpha(u) = -\alpha u^{\alpha-1} \int_0^\infty dy\, g_\alpha(y)\, y^{-2\alpha} \int_1^\infty \frac{dt\, t}{\sqrt{t^{1/\alpha}-1}}\, e^{-t[u/y]^\alpha}$$

$$= -\alpha u^{-(\alpha+1/2)} \int_0^\infty \frac{dy\, g_\alpha(y)}{\sqrt{y}} \int_{(u/y)^\alpha}^\infty \frac{ds\, s^{1-1/(2\alpha)} e^{-s}}{\sqrt{1-(\frac{u}{y})\, s^{-1/\alpha}}}$$

Therefore we obtain

$$F_\alpha(u) = -\frac{1}{(2\pi)^{3/2}\Gamma(1/2)} \frac{d}{du} \int_0^\infty \frac{dy\, g_\alpha(y)}{\sqrt{y}}\, R_\alpha\left(\frac{u^\alpha}{y^\alpha}\right)$$

with

$$R_\alpha(\Theta) = \int_\Theta^\infty \frac{ds\, se^{-s}}{\sqrt{s^{1/\alpha} - \Theta^{1/\alpha}}}$$

In order to evaluate the derivative appearing in the expression of $F_\alpha(u)$ we represent $R_\alpha(\Theta)$ as

$$R_\alpha(\Theta) = 2\alpha \int_\Theta^\infty ds\, s^{2-1/\alpha} e^{-s} (\sqrt{s^{1/\alpha} - \Theta^{1/\alpha}})'$$

$$= 2\alpha \int_\Theta^\infty ds\, e^{-s} (\sqrt{s^{1/\alpha} - \Theta^{1/\alpha}}) \left[\left(\frac{1}{\alpha} - 2 \right) s^{1-2/\alpha} + s^{2-1/\alpha} \right]$$

then

$$R'_\alpha(\Theta) = -\Theta^{1/\alpha-1} \int_\Theta^\infty \frac{ds\, s^{1-1/\alpha} e^{-s}}{\sqrt{s^{1/\alpha} - \Theta^{1/\alpha}}} (1/\alpha - 2 + s)$$

Finally we obtain

$$F_\alpha(u) = \frac{1}{(2\pi)^{3/2}\Gamma(1/2)} \int_0^\infty \frac{dy\, g_\alpha(y)}{y^{3/2}} \int_{\frac{u^\alpha}{y^\alpha}}^\infty \frac{ds (1 - 2\alpha + \alpha s)\, e^{-s}}{s^{1/\alpha - 1} \sqrt{s^{1/\alpha} - (u/y)}} \tag{5.16}$$

Thus we proved the following

Lemma 5.1. If $F_\alpha(u)$ $(u > 0,\ 0 < \alpha < 1)$ is defined by formulas (5.6) and (5.16), then the following identity holds:

$$\int_{\Re^3} d\mathbf{v}\, F_\alpha\left(\frac{|\mathbf{v}|^2}{2}\right) e^{-i\mathbf{k}\cdot\mathbf{v}} = \sum_0^\infty (-1)^n \frac{n+1}{\Gamma(1+n\alpha)} \left(\frac{|\mathbf{k}|^2}{2}\right)^{n\alpha}$$

This yields a unified representation of both exact solutions of the Boltzmann equation (2.1) with $g = 1/(4\pi)$:

$$f_\alpha(\mathbf{v}, t) = e^{-3\lambda_\alpha t} F_\alpha\left(\frac{|\mathbf{v}|^2}{2} e^{-2\lambda_\alpha t}\right)$$

where $\alpha = 1/2,\ 1/3$ and $\lambda_{1/2} = 1/3,\ \lambda_{1/3} = 3/4$. The formula (5.16) clearly shows that both solutions are positive. One can easily verify that the case $\alpha = 1/2$ with $g_{1/2}$ given by (5.12) leads to the function (4.2) found in Section 4.

The integral (5.16) has a nontrivial asymptotics for $u \to 0$. We omit relatively simple calculations leading to the following results:

(1) If $3/4 < \alpha \leqslant 1$, then

$$F_\alpha(0) = \frac{1}{2^{5/2}\pi^2} \Gamma\left(2 - \frac{3}{2\alpha}\right) \int_0^\infty \frac{dy\, g_\alpha(y)}{y^{3/2}}$$

(2) If $0 < \alpha \leqslant 3/4$ and $\alpha \neq 1/2$, then

$$F_\alpha(u) \cong_{u \to 0} \frac{(1-2\alpha)\, u^{-(3-4\alpha)/2}}{\sqrt{2}\, \pi^2 \Gamma(1+2\alpha)} \int_1^\infty \frac{ds}{s^{1/\alpha - 1} \sqrt{s^{1/\alpha} - 1}}$$

where we used the formula

$$\int_0^\infty \frac{dy\, g_\alpha(y)}{y^{2\alpha}} = \frac{2}{\Gamma(1+2\alpha)}$$

which follows from Eqs. (5.3), (5.7).

The intermediate case $\alpha = 1/2$ was studied in Section 4. Thus the function $F_\alpha(u)$ in (5.16) is positive if and only if $0 < \alpha \leqslant 1/2$. In the case $\alpha = 1/3$ we obtain (notation as in (5.13)):

$$F_{1/3}(\mathbf{v}) = \tilde{F}_{1/3}\left(\frac{|\mathbf{v}|^2}{2}\right) \cong_{|\mathbf{v}| \to 0} A\, |\mathbf{v}|^{-5/3}$$
$$A = \frac{2^{-5/3}\sqrt{3}}{\pi^{5/2}} \Gamma\left(\frac{5}{6}\right) \tag{5.17}$$

for the solution (5.1) of the Boltzmann equation.

Let us now consider the asymptotic behavior of $F_\alpha(u)$ as $u \to \infty$. Formula (5.6) shows that

$$g_\alpha \cong_{x \to \infty} \frac{\alpha}{\Gamma(1-\alpha)} x^{-(1+\alpha)}$$

On the other hand, Eq. (5.16) can be written as

$$F_\alpha(u) = \frac{u^{-1/2}}{(2\pi)^{3/2}\,\Gamma(1/2)} \int_0^\infty \frac{dy\, g_\alpha(u/y)}{y^{1/2}} \int_{y^\alpha}^\infty \frac{ds\,(1-2\alpha+\alpha s)\, e^{-s}}{s^{1/\alpha-1}\sqrt{s^{1/\alpha}-y}}$$

Then we obtain

$$F_\alpha(u) \cong_{u\to\infty} B_\alpha u^{-(\alpha+3/2)},$$

$$\begin{aligned} B_\alpha &= \frac{\alpha}{(2\pi)^{3/2}\,\Gamma(1/2)\,\Gamma(1-\alpha)} \int_0^\infty dy\, y^{1/2+\alpha} \int_{y^\alpha}^\infty \frac{ds\,(1-2\alpha+\alpha s)\, e^{-s}}{s^{1/\alpha-1}\sqrt{s^{1/\alpha}-y}} \\ &= \frac{2^{-1/2}(1+\alpha)}{\pi^2\Gamma(1-\alpha)} \int_1^\infty \frac{ds\, s^{-(2+\frac{1}{\alpha})}}{\sqrt{s^{1/\alpha}-1}} \end{aligned}$$

Thus in the case $\alpha = 1/3$ we get (see Eq. (5.13)):

$$F_{1/3}(\mathbf{v}) = \tilde{F}_{1/3}\left(\frac{|\mathbf{v}|^2}{2}\right) \cong_{|\mathbf{v}|\to\infty} B\,|\mathbf{v}|^{-11/3}$$

$$B = \frac{5\cdot 2^{-2/3}}{\sqrt{3}\,\pi^{5/2}}\,\Gamma\left(\frac{5}{6}\right)$$

for the solution (5.1) of the Boltzmann equation.

According to the remark at the end of Section 4, taking the convolution in $\mathfrak{R}^3$ of a solution with an arbitrary Maxwellian produces a one-parameter family of smooth eternal solutions of the Boltzmann equation (2.1).

6. CONCLUDING REMARKS

We have constructed two positive, self-similar solutions of the Boltzmann equation, which turn out to be eternal. They do not have finite energy. By convolution with a Maxwellian we can produce two families of smooth, eternal, positive solution.

We have insisted on the positivity of these solutions, because it is easy to give examples of eternal solutions, which are negative in some nonzero measure set. In particular, the function

$$f(\mathbf{v}, t) = e^{3\lambda t} F(\mathbf{v} e^{\lambda t}) \tag{6.1}$$

where λ is defined by Eq. (2.17) and

$$\int_{\mathfrak{R}^3} d\mathbf{v}\, F(\mathbf{v})\, e^{-i\mathbf{k}\cdot\mathbf{v}} = \left(1+\frac{|\mathbf{k}|^2}{2}\right) e^{-\frac{|\mathbf{k}|^2}{2}}$$

gives an example of eternal solution having finite moments of any order. This example holds true for any scattering function $g(\mu)$ in (2.1). However,

$$F(\mathbf{v}) = (2\pi)^{-3/2}\, e^{-|\mathbf{v}|^2} \left(\frac{5-|\mathbf{v}|^2}{2}\right) \tag{6.2}$$

obviously violates the positivity condition. In some sense, the solution (6.1)–(6.2) describes an asymptotics for $t \to -\infty$ for a general class of eternal solutions with all moments finite. Moreover, the solutions obtained above (generalized to arbitrary Maxwell-type cross-sections) seem to be asymptotic states (as $t \to +\infty$) of initial states with infinite energy. We plan to consider related questions in a subsequent paper.[8]

ACKNOWLEDGMENTS

The research described in the paper was performed in the frame of European TMR (contract ERBFMRXCT970157) and was also partially supported by MURST of Italy and NFR of Sweden. Some results of this paper were briefly reported on by one of authors (A.B.) at the Oberwolfach conference on “Kinetic theory and asymptotic methods” (April 16–20, 2001); we thank all the participants for useful discussions.

REFERENCES

1. A. V. Bobylev, A class of invariant solutions of the Boltzmann equation, *Doklady Akad. Nauk SSSR* **231**:571–574 (1976).
2. M. Ernst, Nonlinear model Boltzmann equations and exact solutions, *Phys. Rep.* **78**:1–171 (1981).
3. A. V. Bobylev, Exact solutions of the nonlinear Boltzmann equation and the theory of relaxation of a maxwellian gas, *Teoret. Mat. Fiz.* **60**:280–310 (1984).
4. M. Barnsley and H. Cornille, On a class of solutions of the Krook–Wu model of the Boltzmann equation, *J. Mat. Phys.* **21**:1176–1193 (1980).

5. H. Grad, Singular and nonuniform limits of solutions of the Boltzmann equation, in *Transport Theory* (Proc. Sympos. Appl. Math., New York, 1967), SIAM-AMS Proc., Vol. I, pp. 269–308 (Amer. Math. Soc., Providence, R.I., 1969).
6. C. Cercignani, A. Frezzotti, and P. Grosfils, The structure of an infinitely strong shock wave, *Phys. Fluids* **11**:2757–2764 (1999).
7. S. Takata, K. Aoki, and C. Cercignani, The velocity distribution function in an infinitely strong shock wave, *Phys. Fluids* **12**:2116–2127 (2000).
8. A. V. Bobylev and C. Cercignani, Self-similar solutions of the Boltzmann equation and their applications, *J. Stat. Phys.* **106**:1039–1072 (2002).
9. C. Villani, *Contribution à l'étude mathématique des équations de Boltzmann et de Landau en théorie cinétique des gaz et des plasmas*, Thèse de l'Université de Paris IX-Dauphine, pp. 284–294 (1998).
10. R. S. Krupp, *A Nonequilibrium Solution of the Fourier Transformed Boltzmann Equation*, submitted to the Department of Aeronautics and Astronautics in partial fullfillment of the requirements for the degree of Master of Science, M.I.T., Cambridge, Mass. (1967).
11. A. V. Bobylev, Exact solutions of the Boltzmann equation, *Doklady Akad. Nauk SSSR* **225**:1296–1299 (1975).
12. M. Krook and T. T. Wu, Formation of Maxwellian tails, *Phys. Rev. Lett.* **36**:1107–1189 (1976).
13. I. S. Gradshtein and I. M. Ryzhik, *Table of Integrals, Series, and Products*, Translated from the Russian (Academic Press, New York, 1965).
14. H. Cabannes, Proof of the conjecture on "eternal" positive solutions for a semi-continuous model of the Boltzmann equation, *C. R. Acad. Sci. Paris Sér. I Math.* **327**:217–222 (1988).
15. H. Cabannes, "Eternal" solutions for a model of the Boltzmann equation, *Math. Models Methods Appl. Sci.* **9**:127–137 (1999).
16. W. Feller *An Introduction to Probability Theory and Its Applications*, Vol. 1 (Wiley, N.Y., 1966).
17. E. Lukacs, *Characteristic Functions* (Griffin, London, 1970).